8 Numerické riešenie lineárnej sústavy

Príklad 1 *Nájdime LU rozklad matice*

$$A = \begin{pmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{pmatrix}.$$

Hľadáme obvyklým spôsobom

$$U^* = \begin{pmatrix} 3 & 1 & 6 \\ 0 & \dots & \dots \\ 0 & \dots & \dots \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ \dots & 1 & 0 \\ \dots & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{pmatrix} \begin{matrix} \\ -(-\mathbf{2}) \cdot I \\ \mathbf{0} \cdot I \end{matrix}$$

$$U^* = \begin{pmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ -\mathbf{2} & 1 & 0 \\ \mathbf{0} & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{pmatrix} \begin{matrix} \\ \\ -(\mathbf{4}) \cdot II \end{matrix}$$

$$U = \begin{pmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ -\mathbf{2} & 1 & 0 \\ \mathbf{0} & \mathbf{4} & 1 \end{pmatrix}$$

A platí

$$L \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ -\mathbf{2} & 1 & 0 \\ \mathbf{0} & \mathbf{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{pmatrix}$$

Príklad 2 *Nájdime LU rozklad matice*

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

Rozklad hľadáme pomocou elementárnych riadkových operácii úpravou na hornotrojuholníkový tvar

$$U^* = \begin{pmatrix} 8 & -6 & 2 \\ 0 & \dots & \dots \\ 0 & \dots & \dots \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ \dots & 1 & 0 \\ \dots & \dots & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{matrix} \\ -\left(-\frac{6}{8}\right) \cdot I = -\left(-\frac{3}{4}\right) \cdot I \\ -\left(\frac{2}{8}\right) \cdot I = -\left(\frac{1}{4}\right) \cdot I \end{matrix}$$

$$U^* = \begin{pmatrix} 8 & -6 & 2 \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & -\frac{5}{2} & \frac{5}{2} \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{4} & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -6 & 2 \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & -\frac{5}{2} & \frac{5}{2} \end{pmatrix} \begin{matrix} \\ \\ -(-1) \cdot II \end{matrix}$$

$$U = \begin{pmatrix} 8 & -6 & 2 \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{4} & -1 & 1 \end{pmatrix}$$

A teda naozaj platí

$$L \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{4} & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 8 & -6 & 2 \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

Príklad 3 *Nájdime LU rozklad matice*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Horno trojuholníkovú maticu U a dolnotrojuhokníkovú maticu L si predpripravíme

$$U^* = \begin{pmatrix} 1 & 2 & 3 \\ 0 & \dots & \dots \\ 0 & \dots & \dots \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ \dots & 1 & 0 \\ \dots & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{matrix} \\ -(\mathbf{4}) \cdot I \\ -(\mathbf{7}) \cdot I \end{matrix}$$

$$U^* = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{4} & 1 & 0 \\ \mathbf{7} & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \begin{matrix} \\ \\ -(\mathbf{2}) \cdot II \end{matrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$$

A teda platí

$$L \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Príklad 4 *Nájdime Choleskyho rozklad matice*

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}.$$

Prvky matice hľadáme pomocou vzťahov

$$\begin{aligned} l_{j,j} &= \sqrt{a_{j,j} - \sum_{k=1}^{j-1} l_{j,k}^2}, \qquad j = 1, 2, \ldots, n, \\ l_{i,j} &= \frac{1}{l_{j,j}} \cdot \left(a_{i,j} - \sum_{k=1}^{j-1} l_{i,k} \cdot l_{j,k} \right) \qquad i = j+1, j+2, \ldots, n. \end{aligned}$$

A teda zrejme platí

$$\begin{aligned} l_{1,1} &= l_{j,j} = \sqrt{a_{1,1}} = \sqrt{4} = 2, \\ l_{2,1} &= l_{i,j} = \frac{a_{2,1}}{l_{1,1}} = \frac{12}{2} = 6, \\ l_{2,2} &= l_{j,j} = \sqrt{a_{2,2} - l_{2,1}^2} = \sqrt{37 - 6^2} = \sqrt{37 - 36} = \sqrt{1} = 1, \\ l_{3,1} &= l_{i,j} = \frac{a_{3,1}}{l_{1,1}} = \frac{-16}{2} = -8, \\ l_{3,2} &= l_{i,j} = \frac{1}{l_{2,2}} \cdot (a_{3,2} - l_{3,1} \cdot l_{2,1}) = \frac{-43 - (-8) \cdot 6}{1} = 5, \\ l_{3,3} &= l_{j,j} = \sqrt{a_{3,3} - \left(l_{3,1}^2 + l_{3,2}^2\right)} = \sqrt{98 - \left((-8)^2 + 5^2\right)} = \sqrt{9} = 3. \end{aligned}$$

Choleskyho rozklad matice A je

$$L \cdot L^T = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}.$$

Príklad 5 *Nájdime Choleskyho rozklad matice*

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}.$$

Využijeme vzťahy

$$\begin{aligned} l_{j,j} &= \sqrt{a_{j,j} - \sum_{k=1}^{j-1} l_{j,k}^2}, \qquad j = 1, 2, \ldots, n, \\ l_{i,j} &= \frac{1}{l_{j,j}} \cdot \left(a_{i,j} - \sum_{k=1}^{j-1} l_{i,k} \cdot l_{j,k} \right), \qquad i = j+1, j+2, \ldots, n, \end{aligned}$$

a teda platí

$$\begin{aligned} l_{1,1} &= l_{j,j} = \sqrt{a_{1,1}} = \sqrt{25} = 5, \\ l_{2,1} &= l_{i,j} = \frac{1}{l_{1,1}} \cdot (a_{2,1}) = \frac{15}{5} = 3, \\ l_{2,2} &= l_{j,j} = \sqrt{a_{2,2} - l_{2,1}^2} = \sqrt{18 - 3^2} = \sqrt{9} = 3, \\ l_{3,1} &= l_{i,j} = \frac{1}{l_{1,1}} \cdot (a_{3,1}) = \frac{-5}{5} = -1, \\ l_{3,2} &= l_{i,j} = \frac{1}{l_{2,2}} \cdot (a_{3,2} - l_{3,1} \cdot l_{2,1}) = \frac{1}{3} \cdot (0 - (-1) \cdot 3) = 1, \\ l_{3,3} &= l_{j,j} = \sqrt{a_{3,3} - \left(l_{3,1}^2 + l_{3,2}^2\right)} = \sqrt{11 - \left((-1)^2 + 1^2\right)} = 3. \end{aligned}$$

Zrejme platí

$$L \cdot L^T = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

Príklad 6 *Nájdime Choleskyho rozklad matice*

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{pmatrix}.$$

Prvky matice hľadáme pomocou vzťahov

$$\begin{aligned} l_{j,j} &= \sqrt{a_{j,j} - \sum_{k=1}^{j-1} l_{j,k}^2}, \qquad j = 1, 2, \ldots, n, \\ l_{i,j} &= \frac{1}{l_{j,j}} \cdot \left(a_{i,j} - \sum_{k=1}^{j-1} l_{i,k} \cdot l_{j,k} \right), \qquad i = j+1, j+2, \ldots, n. \end{aligned}$$

A teda

$$\begin{aligned}
l_{1,1} &= l_{j,j} = \sqrt{a_{1,1}} = \sqrt{1} = 1, \\
l_{2,1} &= l_{i,j} = \frac{1}{l_{1,1}} \cdot (a_{2,1}) = \frac{-1}{1} = -1, \\
l_{2,2} &= l_{j,j} = \sqrt{a_{2,2} - l_{2,1}^2} = \sqrt{5 - (-1)^2} = \sqrt{4} = 2, \\
l_{3,1} &= l_{i,j} = \frac{1}{l_{1,1}} \cdot (a_{3,1}) = \frac{2}{1} = 2, \\
l_{3,2} &= l_{i,j} = \frac{1}{l_{2,2}} \cdot (a_{3,2} - l_{3,1} \cdot l_{2,1}) = \frac{1}{2} \cdot (-4 - 2 \cdot (-1)) = -1, \\
l_{3,3} &= l_{j,j} = \sqrt{a_{3,3} - \left(l_{3,1}^2 + l_{3,2}^2\right)} = \sqrt{6 - \left(2^2 + (-1)^2\right)} = \sqrt{1} = 1.
\end{aligned}$$

Čiže dolno trojuholníková matica je

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

a platí

$$L \cdot L^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{pmatrix}.$$

Príklad 7 *Nájdime Choleskyho rozklad matice*

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{pmatrix}.$$

Prvky matice hľadáme pomocou vzťahov

$$\begin{aligned}
l_{j,j} &= \sqrt{a_{j,j} - \sum_{k=1}^{j-1} l_{j,k}^2}, \qquad j = 1, 2, \ldots, n, \\
l_{i,j} &= \frac{1}{l_{j,j}} \cdot \left(a_{i,j} - \sum_{k=1}^{j-1} l_{i,k} \cdot l_{j,k}\right), \qquad i = j+1, j+2, \ldots, n.
\end{aligned}$$

A teda

$$\begin{aligned}
l_{1,1} &= l_{j,j} = \sqrt{a_{j,j}} = \sqrt{1} = 1, \\
l_{2,1} &= l_{i,j} = \frac{1}{l_{2,2}} \cdot (a_{2,1}) = \frac{-1}{1} = -1, \\
l_{2,2} &= l_{j,j} = \sqrt{a_{2,2} - l_{2,1}^2} = \sqrt{5 - (-1)^2} = \sqrt{4} = 2, \\
l_{3,1} &= l_{i,j} = \frac{1}{l_{1,1}} \cdot (a_{3,1}) = \frac{2}{1} = 2, \\
l_{3,2} &= l_{i,j} = \frac{1}{l_{2,2}} \cdot (a_{3,2} - l_{3,1} \cdot l_{2,1}) = \frac{1}{2} \cdot (-4 - 2 \cdot (-1)) = \frac{-2}{2} = -1, \\
l_{3,3} &= \sqrt{a_{3,3} - \left(l_{3,1}^2 + l_{3,2}^2\right)} = \sqrt{6 - \left(2^2 + (-1)^2\right)} = \sqrt{1} = 1.
\end{aligned}$$

A naozaj platí

$$L \cdot L^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{pmatrix}.$$

Príklad 8 *Jacobiho a Gaussovou-Seidelovou metódou s presnosťou* $\varepsilon = 0.01$ *riešme sústavu*

$$\begin{aligned}
5x_1 + x_2 + 2x_3 &= 1, \\
x_1 + 4x_2 + x_3 &= 2, \\
2x_1 + 2x_2 + 5x_3 &= 3.
\end{aligned}$$

Matica sústavy

$$\begin{pmatrix} 5 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 2 & 5 \end{pmatrix}$$

je riadkovo diagonálne dominantná

$$\begin{aligned}
|5| &> |1| + |2|, \\
|4| &> |1| + |1|, \\
|5| &> |2| + |2|,
\end{aligned}$$

a preto bude iteračná postupnosť

$$\begin{aligned}
x_1^{(k+1)} &= \left(1 - x_2^{(k)} - 2x_3^{(k)}\right)/5, \\
x_2^{(k+1)} &= \left(2 - x_1^{(k)} - x_3^{(k)}\right)/4, \\
x_3^{(k+1)} &= \left(3 - 2x_1^{(k)} - 2x_2^{(k)}\right)/5,
\end{aligned}$$

bude konvergovať k presnému riešeniu pre akúkoľvek ľubovoľnú počiatočnú aproximáciu (aj napr. $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)^T = (0,0,0)^T$*)*

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	0.2000	0.5000	0.6000
2	−0.1400	0.3000	0.3200
3	0.0120	0.4550	0.5360
4	−0.1054	0.3630	0.4132
5	−0.0379	0.4230	0.4970
6	−0.0834	0.3852	0.4459
7	−0.0554	0.4094	0.4793
8	−0.0736	0.3940	0.4584
9	−0.0622	0.4038	0.4718
10	−0.0695	0.3976	0.4634

Nakoľko platí

$$\begin{aligned}\left|x_1^{(10)} - x_1^{(9)}\right| &= |-0.0695 + 0.0622| = 0.0073 < 0.01 = \varepsilon,\\ \left|x_2^{(10)} - x_2^{(9)}\right| &= |0.3976 - 0.4038| = 0.0062 < 0.01 = \varepsilon,\\ \left|x_3^{(10)} - x_3^{(9)}\right| &= |0.4634 - 0.4718| = 0.0084 < 0.01 = \varepsilon,\end{aligned}$$

a teda približné riešenie je

$$\mathbf{x}^{(10)} = \begin{pmatrix} 0.07 \\ 0.40 \\ 0.46 \end{pmatrix}.$$

Iteračná postupnosť Gaussovej-Seideolovej metódy je

$$\begin{aligned}x_1^{(k+1)} &= \left(1 - x_2^{(k)} - 2x_3^{(k)}\right)/5,\\ x_2^{(k+1)} &= \left(2 - x_1^{(k+1)} - x_3^{(k)}\right)/4,\\ x_3^{(k+1)} &= \left(3 - 2x_1^{(k+1)} - 2x_2^{(k+1)}\right)/5,\end{aligned}$$

a pre tú istú počiatočnú aproximáciu máme

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	0.2000	0.4500	0.3400
2	−0.0260	0.4215	0.4418
3	−0.0610	0.4048	0.4625
4	−0.0660	0.4009	0.4660

a keďže platí

$$\begin{aligned} \left|x_1^{(4)} - x_1^{(3)}\right| &= |-0.0660 + 0.0610| = 0.0050 < 0.01 = \varepsilon, \\ \left|x_2^{(4)} - x_2^{(3)}\right| &= |0.4009 - 0.4048| = 0.0039 < 0.01 = \varepsilon, \\ \left|x_3^{(4)} - x_3^{(3)}\right| &= |0.4660 - 0.4625| = 0.0035 < 0.01 = \varepsilon, \end{aligned}$$

približné riešenie je

$$\mathbf{x}^{(4)} = \begin{pmatrix} 0.07 \\ 0.40 \\ 0.47 \end{pmatrix}.$$

Príklad 9 *Jacobiho a Gaussovou-Seidelovou metódou s presnosťou* $\varepsilon = 0.01$ *riešme sústavu*

$$\begin{aligned} 9x_1 + 2x_2 + 4x_3 &= 20, \\ x_1 + 10x_2 + 4x_3 &= 6, \\ 2x_1 - 4x_2 + 10x_3 &= -15. \end{aligned}$$

Matica sústavy

$$\begin{pmatrix} 9 & 2 & 4 \\ 1 & 10 & 4 \\ 2 & -4 & 10 \end{pmatrix}$$

je riadkovo diagonjálne dominantná, keďže platí

$$\begin{aligned} |9| &> |2| + |4|, \\ |10| &> |1| + |4|, \\ |10| &> |2| + |-4|, \end{aligned}$$

a teda iteračná postupnosť Jacobiho metódy

$$\begin{aligned} x_1^{(k+1)} &= \left(20 - 2x_2^{(k)} - 4x_3^{(k)}\right)/9, \\ x_2^{(k+1)} &= \left(6 - x_1^{(k)} - 4x_3^{(k)}\right)/10, \\ x_3^{(k+1)} &= \left(-15 - 2x_1^{(k)} + 4x_2^{(k)}\right)/10 \end{aligned}$$

konverguje pre ľubovovoľnú počiatočnú aproximáciu, čiže i pre $\mathbf{x}^{(0)} = (0.0000, 0.0000, 0.0000)^T$. *Výsledky zapíšeme do tabuľky*

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	2.2222	0.6000	−1.5000
2	2.7556	0.9778	−1.7044
3	2.7625	1.0062	−1.6600
4	2.7364	0.9878	−1.6500
5	2.7361	0.9864	−1.6522

Iteračnú postupnosť ukončíme, nakoľko platí

$$\begin{aligned}\left|x_1^{(5)}-x_1^{(4)}\right| &= |2.7361-2.7364| = 0.0003 < 0.01 = \varepsilon,\\ \left|x_2^{(5)}-x_2^{(4)}\right| &= |0.9864-0.9878| = 0.0014 < 0.01 = \varepsilon,\\ \left|x_3^{(5)}-x_3^{(4)}\right| &= |-1.6522+1.6500| = 0.0022 < 0.01 = \varepsilon\end{aligned}$$

a za približné riešenie považujeme

$$\mathbf{x}^{(5)} = \begin{pmatrix} 2.74 \\ 0.99 \\ -1.65 \end{pmatrix}.$$

A iteračná postupnosť Gaussovej-Seidelovej metódy

$$\begin{aligned}x_1^{(k+1)} &= \left(20-2x_2^{(k)}-4x_3^{(k)}\right)/9,\\ x_2^{(k+1)} &= \left(6-x_1^{(k+1)}-4x_3^{(k)}\right)/10,\\ x_3^{(k+1)} &= \left(-15-2x_1^{(k+1)}+4x_2^{(k+1)}\right)/10\end{aligned}$$

konverguje pre ľubovovoľnú počiatočnú aproximáciu, čiže i pre $\mathbf{x}^{(0)} = (0.0000, 0.0000, 0.0000)^T$. *Výsledky zapíšeme do tabuľky*

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	2.2222	0.3778	−1.7933
2	2.9353	1.0238	−1.6775
3	2.7403	0.9970	−1.6493
4	2.7337	0.9863	−1.6522
5	2.7373	0.9871	−1.6526

nakoľko platí

$$\begin{aligned}\left|x_1^{(5)}-x_1^{(4)}\right| &= |2.7373-2.7337| = 0.0036 < 0.01 = \varepsilon,\\ \left|x_2^{(5)}-x_2^{(4)}\right| &= |0.9871-0.9863| = 0.0008 < 0.01 = \varepsilon,\\ \left|x_3^{(5)}-x_3^{(4)}\right| &= |-1.6526+1.6522| = 0.0004 < 0.01 = \varepsilon\end{aligned}$$

a za približné riešenie považujeme

$$\mathbf{x}^{(5)} = \begin{pmatrix} 2.74 \\ 0.99 \\ -1.65 \end{pmatrix}.$$

Príklad 10 *Jacobiho a Gaussovou-Seidelovou metódou s presnosťou* $\varepsilon = 0.01$ *riešme sústavu*

$$\begin{aligned} 9x_1 + x_2 + x_3 &= 10, \\ 2x_1 + 10x_2 + 3x_3 &= 19, \\ 3x_1 + 4x_2 + 11x_3 &= 0. \end{aligned}$$

Overíme, či je matica sústavy

$$\begin{pmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{pmatrix}$$

diagonálne (riadkovo) dominantná

$$\begin{aligned} |9| &> |1| + |1|, \\ |10| &> |2| + |3|, \\ |11| &> |3| + |4|. \end{aligned}$$

Nakoľko je matica diag. dominantná, bude aj iteračná postupnosť Jacobiho metódy

$$\begin{aligned} x_1^{(k+1)} &= \left(10 - x_2^{(k)} - x_3^{(k)}\right)/9, \\ x_2^{(k+1)} &= \left(19 - 2x_1^{(k)} - 3x_3^{(k)}\right)/10, \\ x_3^{(k+1)} &= \left(0 - 3x_1^{(k)} - 4x_2^{(k)}\right)/11, \end{aligned}$$

bude konvergovať pre ľubovoľnú počiatočnú aproximáciu, teda aj pre $\mathbf{x}^{(0)} = (0.0000, 0.0000, 0.0000)^T$ *a postupné výsledky zaúpíšeme do tabuľky*

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	1.1111	1.9000	0.0000
2	0.9000	1.6778	−0.9939
3	1.0351	2.0182	−0.8556
4	0.9819	1.9496	−1.0162
5	1.0074	2.0085	−0.9768
6	0.9965	1.9915	−1.0051
7	1.0015	2.0022	−0.9960
8	0.9993	1.9985	−1.0012

Keďže platí

$$\max \begin{pmatrix} \left| x_1^{(8)} - x_1^{(7)} \right| \\ \left| x_2^{(8)} - x_2^{(7)} \right| \\ \left| x_3^{(8)} - x_3^{(7)} \right| \end{pmatrix} = \max \begin{pmatrix} |0.9993 - 1.0015| \\ |1.9985 - 2.0022| \\ |-1.0012 + 0.9960| \end{pmatrix} = \max \begin{pmatrix} 0.0022 \\ 0.0037 \\ 0.0052 \end{pmatrix} < 0.01 = \varepsilon,$$

za približné riešenie považujeme vektor

$$\mathbf{x}^{(8)} = \begin{pmatrix} 1.00 \\ 2.00 \\ -1.00 \end{pmatrix}.$$

Iteračná postupnosť Gauss-Seidelovej metódy

$$\begin{aligned} x_1^{(k+1)} &= \left(10 - x_2^{(k)} - x_3^{(k)}\right)/9, \\ x_2^{(k+1)} &= \left(19 - 2x_1^{(k+1)} - 3x_3^{(k)}\right)/10, \\ x_3^{(k+1)} &= \left(0 - 3x_1^{(k+1)} - 4x_2^{(k+1)}\right)/11, \end{aligned}$$

a opäť

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	1.1111	1.6778	−0.9131
2	1.0262	1.9687	−0.9958
3	1.0030	1.9981	−1.0001
4	1.0002	2.0000	−1.0001

Keďže platí

$$\max\begin{pmatrix} \begin{vmatrix} x_1^{(4)} - x_1^{(3)} \\ x_2^{(4)} - x_2^{(3)} \\ x_3^{(4)} - x_3^{(3)} \end{vmatrix} \end{pmatrix} = \max\begin{pmatrix} |1.0002 - 1.0030| \\ |2.0000 - 1.9981| \\ |-1.0001 + 1.0001| \end{pmatrix} = \max\begin{pmatrix} 0.0028 \\ 0.0019 \\ 0.0000 \end{pmatrix} < 0.01 = \varepsilon,$$

za približné riešenie považujeme vektor

$$\mathbf{x}^{(4)} = \begin{pmatrix} 1.00 \\ 2.00 \\ -1.00 \end{pmatrix}.$$